Hardness of Approximation

Irit Dinur
Weizmann Institute of Science

Approximation

It’s in our nature

- Grades for students
- Decisions about accepting a paper
- Medical decisions
- Political & economic decisions

It’s frustrating

- Because of the gap

It’s sometimes (NP-)hard, so

- We shouldn’t be blamed for this gap
- We should understand which instances are easy, and which are hard

Approximation

- Throughout 70's-80's: many optimization problems discovered to be NP-hard
- Natural to seek approximate solutions. (Almost no known lower bounds)

OPTIMIZATION

- **Max-LINEQ**: satisfy the largest number of linear equations.
- **Max-3-col**: color the vertices with 3 colors, maximizing number of two-colored edges.
- **Max-3SAT**: assign Boolean variables in a 3CNF formula, maximizing number of satisfied clauses.
- All of these problems are NP-hard (yes, even Max-LINEQ!)

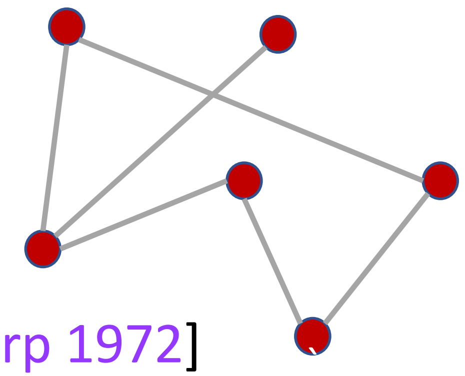
APPROXIMATION

- satisfy at least $\alpha \cdot OPT$ of the equations.
- satisfy at least $\alpha \cdot OPT$ of the edge constraints.
- satisfy at least $\alpha \cdot OPT$ of the clauses.
- Complexity depends on the problem, and on α

NP hard problem: 3-COLORABILITY

A graph G=(V,E) is in 3-COLORABILITY if there a vertex-coloring with 3 colors, where every edge sees different colors

$$G = (V, E)$$



Theorem 1: 3-COLORABILITY is NP-complete [Karp 1972]

Approximate? We can aim to approximate OPT(G) = maximal fraction of edges sat are satisfied by a 3-coloring.

An $(1-\epsilon)$ – approximation algorithm outputs a value α

$$(1-\epsilon) \cdot OPT(G) \leq \alpha \leq OPT(G)$$

Theorem 2: 3-COLORABILITY is NP-hard to approximate (i.e. an approximation algorithm implies P=NP)

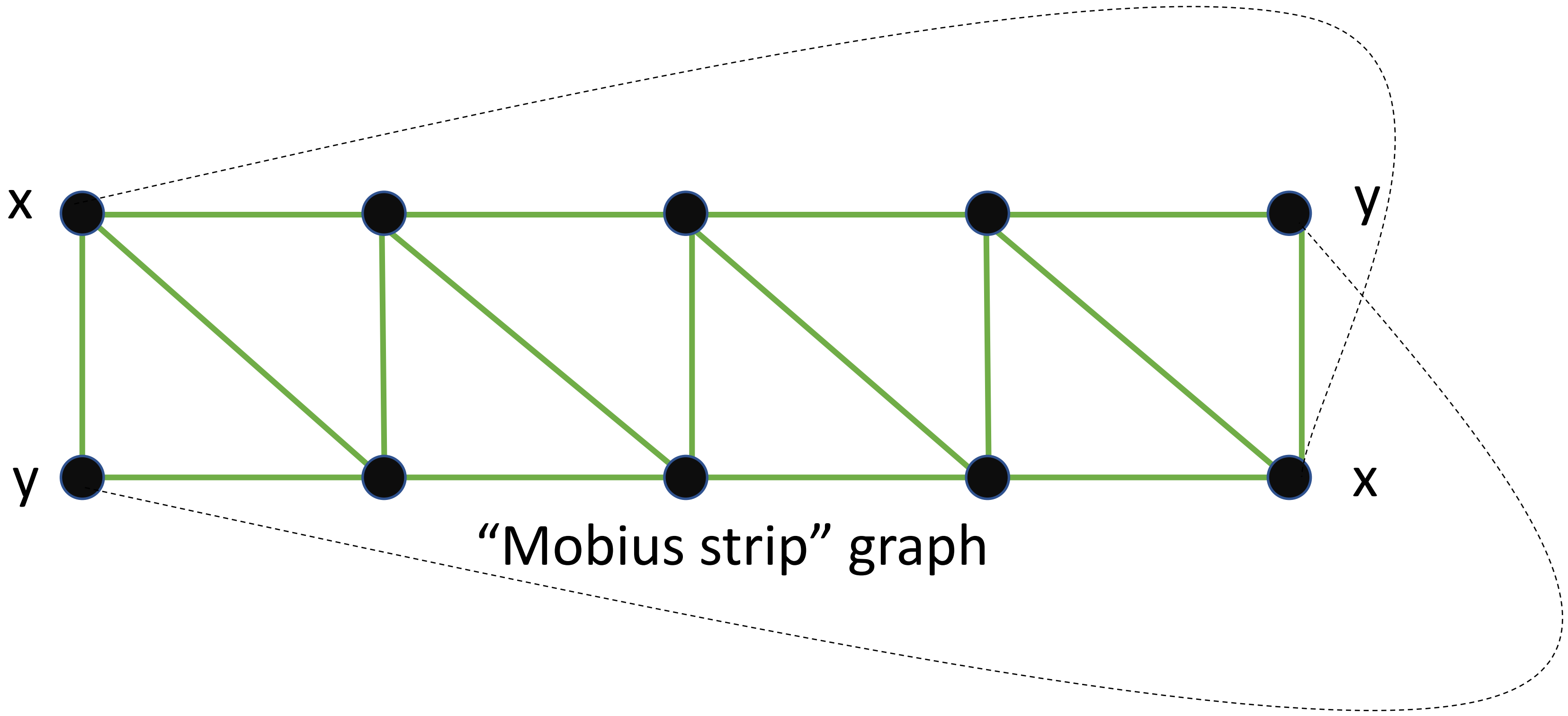
[PCP theorem, Arora-Safra, Arora-Lund-Motwani-Sudan-Szegedy 1991]

PCP theorem

Theorem:
There is a poly-time algorithm R converting G to H, such that
G is 3-colorable ➔ H is 3-colorable
G is not 3-colorable ➔ H has value < $1-\epsilon$
(every coloring fails at least ϵ edges)

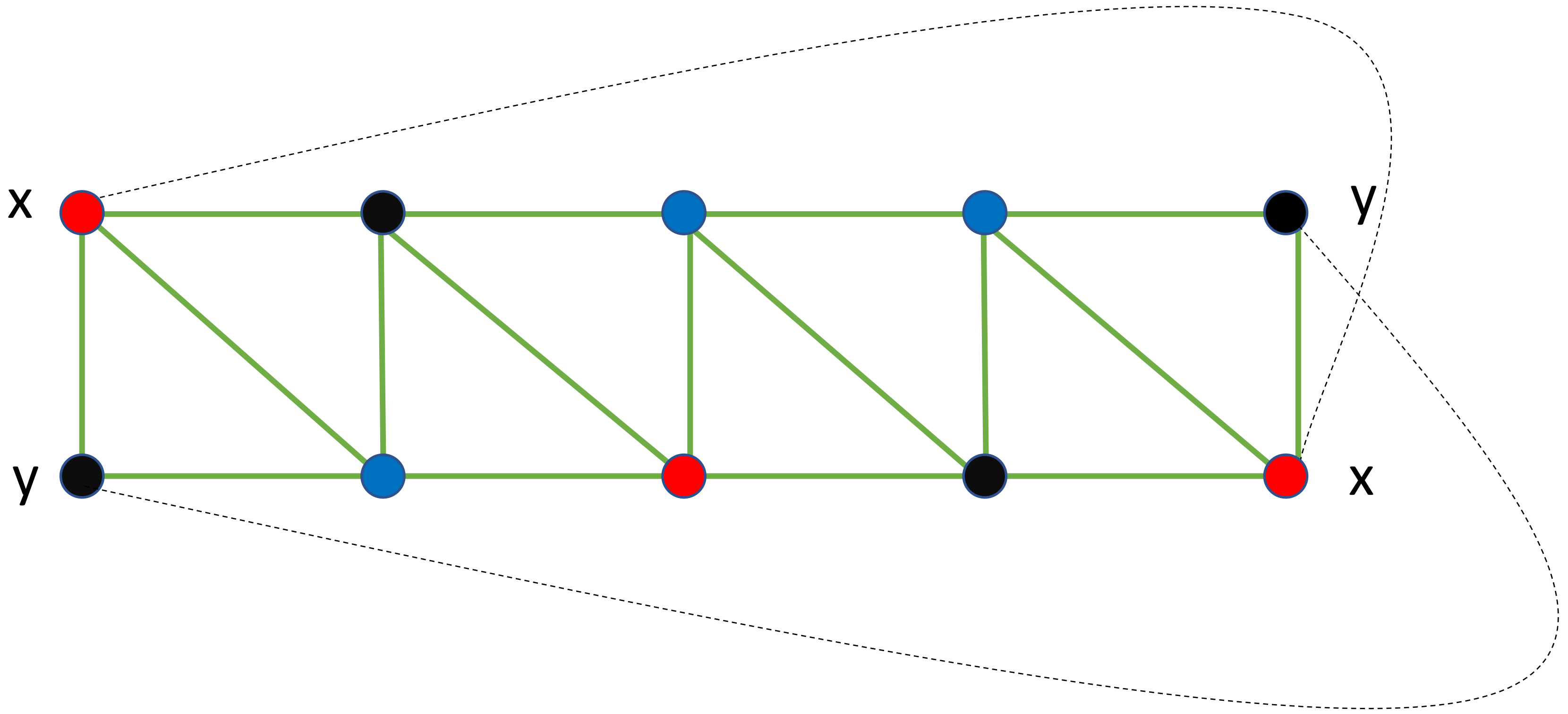
Corollary: This rules out $1-\epsilon$ approximation for 3-COLORABILITY
Proof:
Assume Max-3COL had a $1-\epsilon$ approximation algorithm A.
We construct a polytime algorithm for 3-COLORABILITY (thus P=NP, a contradiction).
Given a graph G, run the algorithm R above to get H, and then run A on H.
If $\mathrm{A}(\mathrm{R}(\mathrm{G})) = \mathrm{A}(\mathrm{H}) \geq 1-\epsilon$, say YES, otherwise say NO.

Graphs don't typically have a 3-COLORABILITY gap



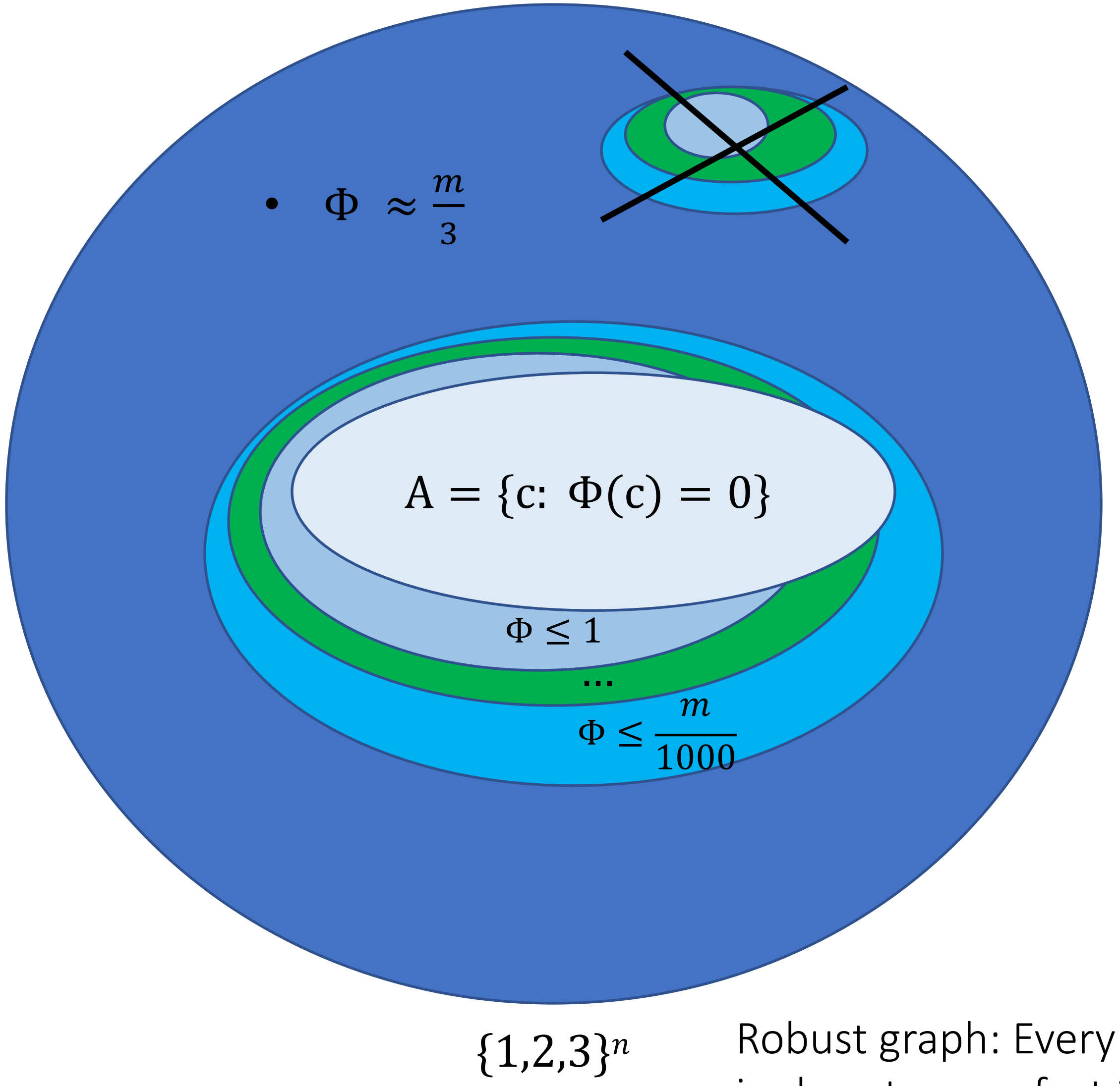
"Mobius strip" graph

Graphs don't typically have a 3-COLORABILITY gap



G is almost 3 colorable: can color vertices so that only one edge fails

Landscape of a robust graph



Let G be a graph. For each $c \in \{1,2,3\}^n$

Denote $\Phi(c) =$

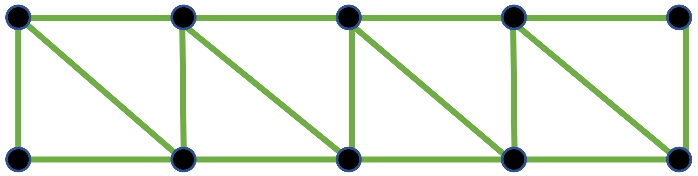
number of edges failing on coloring c.

Let $A \subseteq \{1,2,3\}^n$ be defined by

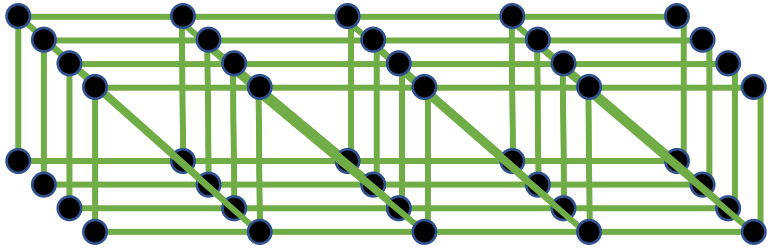
$$A = \Phi^{-1}(0)$$

Robust graph: Every nearly-satisfying 3-coloring is close to a perfect 3-coloring

PCP as a robustification algorithm



$G \to H$



- Approach: Embed G in robust graph H
- PCP theorem: There is an algorithm for transforming G to H s.t.
  - G is 3-colorable iff H is 3-colorable
  - H is c-robust
- In particular: if G isn't 3-colorable, then every 3-coloring for H violates c fraction of edges

How easily can proofs be checked?

The Riemann Hypothesis is true (12th Revision)

By

Ayror Sappen

 # Pages to follow: 15783

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