Geometric Complexity Theory

Christian Ikenmeyer

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https://www.dcs.warwick.ac.uk/~u2270030

Agenda

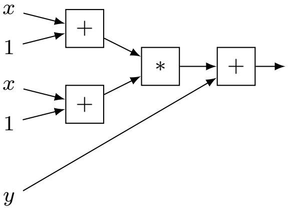
1. Algebraic Complexity Theory
2. Closures, border complexity, and de-bordering
3. Group actions and highest weights

Agenda

Recall: A p-family is a sequence of polynomials with:

- number of variables is polynomially bounded,
- degree is polynomially bounded.

Example: $(\mathrm{per}_m)_{m\in\mathbb{N}}$, $\mathrm{per}_m = \sum_{\pi\in\mathfrak{S}_m}\prod_{i=1}^{m} x_{i,\pi(i)}$.



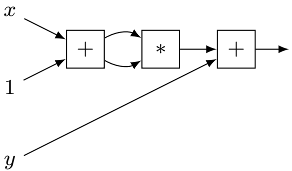
VF

polyn. bounded formula compl.

$$\det\begin{pmatrix} x+1 & y \\ -1 & x+1 \end{pmatrix}$$

VBP

polyn. bounded determinantal compl. (dc)



VP

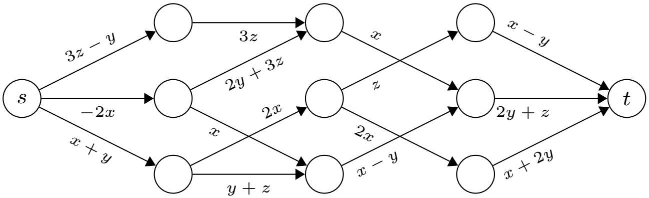
polyn. bounded circuit compl.

Valiant's conjectures:

- $(\mathrm{per}) \notin$ **VF**
- $(\mathrm{per}) \notin$ **VBP**, equivalently $\big(\mathrm{dc}(\mathrm{per}_m)\big)_{m\in\mathbb{N}}$ is not polynomially bounded.
- $(\mathrm{per}) \notin$ **VP**

Note: $\mathrm{dc}(\mathrm{per}_m)$ does not involve any combinatorics of circuits!

Algebraic Branching Programs (ABP)

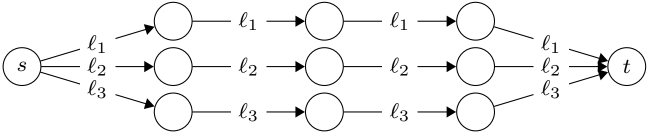


- All edge labels homogeneous linear.
- Computes $\sum_{s\text{-}t\text{-path } p} \prod_{\text{edge } e \in p} \text{label}(e)$
- Computes only homogeneous degree d polynomials.
- $\mathrm{w}(p)$ is the smallest width of an ABP computing p.
- Theorem: $\mathrm{dc}(p)$ and $\mathrm{w}(p)$ are polynomially related.

Corollary (discussed in the Algebraic Complexity Theory sessions)

"$(\mathrm{per}) \notin$ **VBP**" is equivalent to "$\mathrm{w}(\mathrm{per})$ is not polynomially bounded".

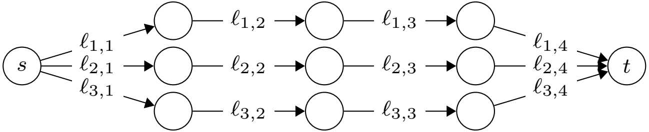
For a homogeneous degree d polynomial p the **Waring rank** $\mathrm{WR}(p)$ is defined as the smallest r such that there exist homogeneous linear ℓ_i such that $p = \sum_{i=1}^{r}(\ell_i)^d$. Not completely obvious at first: $\mathrm{WR}(p)$ is always finite.



$$p = \begin{pmatrix}\ell_1 & \ell_2 & \cdots & \ell_r\end{pmatrix} \begin{pmatrix}\ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r\end{pmatrix} \begin{pmatrix}\ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r\end{pmatrix} \cdots \begin{pmatrix}\ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r\end{pmatrix} \begin{pmatrix}\ell_1 \\ \vdots \\ \ell_r\end{pmatrix}$$

VWaring := set of p-families with polynomially bounded Waring rank.

The **Chow rank** is the smallest r such that there exist homogeneous linear $\ell_{i,j}$ such that $p = \sum_{i=1}^{r}\prod_{j=1}^{d}\ell_{i,j}$.



$$p = \begin{pmatrix}\ell_{1,1} & \ell_{2,1} & \cdots & \ell_{r,1}\end{pmatrix} \begin{pmatrix}\ell_{1,2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_{r,2}\end{pmatrix} \begin{pmatrix}\ell_{1,3} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_{r,3}\end{pmatrix} \cdots \begin{pmatrix}\ell_{1,d-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_{r,d-1}\end{pmatrix} \begin{pmatrix}\ell_{1,d} \\ \vdots \\ \ell_{r,d}\end{pmatrix}$$

VChow := set of p-families with polynomially bounded Chow rank.

For a homogeneous degree d polynomial p the **Waring rank** $\mathrm{WR}(p)$ is defined as the smallest r such that $\exists$ linear forms with

$$p = \begin{pmatrix} \ell_1 & \ell_2 & \cdots & \ell_r \end{pmatrix} \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \cdots \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_r \end{pmatrix} \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_r \end{pmatrix}$$

For a homogeneous degree d polynomial p the **Chow rank** $\mathrm{CR}(p)$ is defined as the smallest r such that $\exists$ linear forms with

$$p = \begin{pmatrix} \ell_{1,1} & \ell_{2,1} & \cdots & \ell_{r,1} \end{pmatrix} \begin{pmatrix} \ell_{1,2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_{r,2} \end{pmatrix} \begin{pmatrix} \ell_{1,3} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_{r,3} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,d-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ell_{r,d-1} \end{pmatrix} \begin{pmatrix} \ell_{1,d} \\ \vdots \\ \ell_{r,d} \end{pmatrix}$$

For a hom. degree d polynomial p the **ABP width complexity** $\mathrm{w}(p)$ is def. as the smallest r such that $\exists$ linear forms with

$$p = \begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

$$p = \begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

Equivalent to $\mathrm{w}(p)$ up to polynomial blowup:

Trace of iterated matrix product:

$$p = \operatorname{tr}\left(\begin{pmatrix} \ell_{1,1,1} & \cdots & \ell_{1,r,1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,1} & \cdots & \ell_{r,r,1} \end{pmatrix} \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1,d} & \cdots & \ell_{1,r,d} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d} & \cdots & \ell_{r,r,d} \end{pmatrix} \right)$$

Trace of matrix power:

$$p = \operatorname{tr}\left(\begin{pmatrix} \ell_{1,1} & \cdots & \ell_{1,r} \\ \vdots & \ddots & \vdots \\ \ell_{r,1} & \cdots & \ell_{r,r} \end{pmatrix}^{d} \right)$$

Newton's identities, determinant, and traces of matrix powers

Power sum: $p_k = x_1^k + \cdots + x_n^k$, $\qquad q_k := (-1)^{k-1} p_k$

Elementary symmetric polynomial: $e_k = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} x_{S_1} x_{S_2} \cdots x_{S_k}$. $\qquad$ e.g. for $n = 3$: $e_2 = x_1x_2 + x_1x_3 + x_2x_3$.

In particular: $e_n = x_1 x_2 \cdots x_n$.

Newton identities:

$$k \cdot e_k = \sum_{i=1}^{k} e_{k-i} \cdot q_i.$$

This works as well for matrices: $M := \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}$

- $P_k := \mathrm{tr}(M^k)$, $\qquad Q_k := (-1)^{k-1} P_k$.
- E_k the degree k coefficient of the characteristic polynomial of M, in particular $E_n = \det(M)$

Newton identities:

$$k \cdot E_k = \sum_{i=1}^{k} E_{k-i} \cdot Q_i.$$

Over char 0 this gives a homogeneous ABP for $\det_n$ of width $O(n^4)$: First, compute E_1, then E_2, and so no.

The determinant is computed from a small number of natural building blocks Q_k, who on their own allow homogeneous computation that captures exactly **VBP**.

- Noncommutative homogeneous polynomials are called **tensors**.
- We write $x \otimes y$ for the product of non-commuting variables.
- Let $\bigotimes^\bullet \mathbb{C}^n$ denote the ring of polynomials in noncommuting variables $x_1, \dots, x_n$.
- Making the variables commute is a surjective ringhomomorphism $\bigotimes^\bullet \mathbb{C}^n \to \mathbb{C}[x_1, \dots, x_n]$.
- WR, CR, w, etc have noncommutative analogues, $\mathrm{WR}_\otimes$, $\mathrm{CR}_\otimes$, $\mathrm{w}_\otimes$.
- $\mathrm{WR}_\otimes$ is only finite for symmetric tensors, and there it equals WR.
- $\mathrm{CR}_\otimes$ is also known as **tensor rank** R.
- $\mathrm{w}_\otimes$ has already been studied by Nisan in 1990.
- The matrix multiplications tensor is homogeneous of degree 3:

$M_n := \sum_{i,j,k=1}^n x_{i,j} \otimes x_{j,k} \otimes x_{k,i}$. $\qquad sM_n := \sum_{i,j,k=1}^n x_{i,j} x_{j,k} x_{k,i}$.

- We have $\mathrm{w}_\otimes(M_n) = n^2$, lower bound by Nisan. Also $\mathrm{w}(sM_n) = n^2$.
- Strassen 1969: $R(M_2) = 7$, which can recursively be used to multiply two $n \times n$ matrices in time $O(n^{\log_2 7}) = O(n^{2.81})$.

The matrix multiplication exponent:

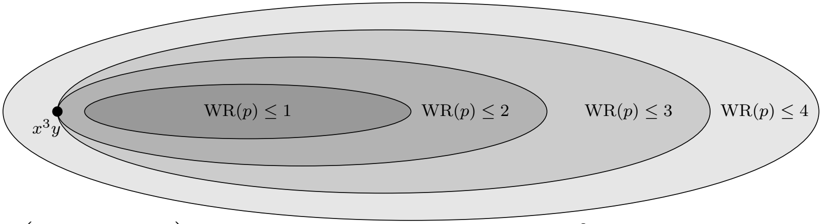
$$\begin{aligned} \omega &= \lim_{n\to\infty} \log_n(R(M_n)) \\ &= \lim_{n\to\infty} \log_n(\mathrm{WR}(sM_n)) \\ &= \lim_{n\to\infty} \log_n(\mathrm{CR}(sM_n)) \end{aligned}$$

[Chiantini-I-Landsberg-Ottaviani 2017]

Summary part "Algebraic Complexity Theory"

- Algebraic computation can be made homogeneous.
- For fast matrix multiplication, computation is homogeneous of degree 3.
- Circuits and formulas are difficult to understand, partly because of their combinatorial structure, so determinantal complexity was welcome.
- The determinant consists of homogeneous building blocks: Iterated matrix multiplication. They allow a homogeneous formulation of **VBP**.

$12x^3y = (x+y)^4 + i^3(x+iy)^4 + i^2(x+i^2y)^4 + i(x+i^3y)^4$, hence $\mathrm{WR}(x^3y) \le 4$. In fact, $\mathrm{WR}(x^{d-1}y) = d$.



$\frac{1}{\varepsilon}\left((x+\varepsilon y)^4 - x^4\right) = 4x^3y + \varepsilon(6x^2y^2 + 4\varepsilon xy^3 + \varepsilon^2 y^4) \xrightarrow{\varepsilon \to 0} 4x^3y$

The **border Waring rank** $\underline{\mathrm{WR}}(p)$ is defined as the smallest r such that p can be approximated arbitrarily closely by polynomials of Waring rank $\le r$. For example, $\underline{\mathrm{WR}}(x^3y) \le 2$.

Analogously define $\underline{\mathrm{CR}}$, $\underline{\mathrm{w}}$, etc

Theorem (works in high generality, $\underline{\mathrm{CR}}$, $\underline{\mathrm{w}}$, etc)

Let $V = \mathbb{C}[x_1, \ldots, x_n]_d$. Zariski closure and Euclidean closure coincide:

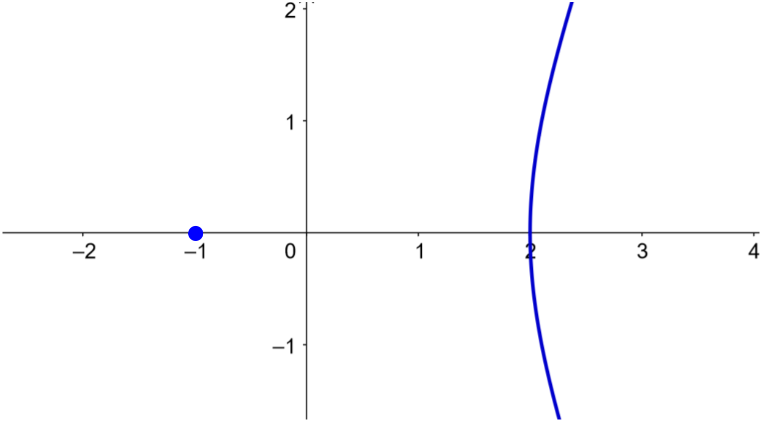
$$\{p \in V \mid \underline{\mathrm{WR}}(p) \le r\} = \overline{\{p \in V \mid \mathrm{WR}(p) \le r\}}^{\mathbb{C}} = \overline{\{p \in V \mid \mathrm{WR}(p) \le r\}}^{\mathsf{Zar}}.$$

In other words: $\{p \in V \mid \underline{\mathrm{WR}}(p)\}$ is an algebraic variety.

(Proof: Chevalley's theorem (images of constructible sets are constructible), and the Zariski closure and Euclidean closure coincide for constructible sets)

Equivalent: Points can be separated from varieties via **polynomials**

$\underline{\mathrm{WR}}(q) > r$ iff there exists a homogeneous polynomial Δ with • $\Delta(p) = 0$ for all p with $\underline{\mathrm{WR}}(p) \le r$ and • $\Delta(q) \ne 0$.



$$x^3 - 3x - 2 - y^2 = 0$$

Example

$\mathbb{A} := \mathbb{C}[x,y]_2 = \langle x^2, xy, y^2 \rangle$.

Every element in $\mathbb{A}$ can be represented as $ax^2 + bxy + cy^2$.

- $X := \{p \in \mathbb{A} \mid \exists \alpha, \beta \in \mathbb{C} : p = (\alpha x + \beta y)^2\}$ set of Waring rank 1 polynomials
- $p \in X$ iff $\Delta(p) = b^2 - 4ac = 0$.
- To prove $\mathrm{WR}(p) \geq 2$ we compute $\Delta(p) \neq 0$

We will study such functions Δ later.

Using metapolynomials for lower bounds:

- Many algebraic complexity lower bounds use this technique, for example all techniques based on matrix ranks.
- For border complexity, the metapolynomials must exist!
- For non-border complexity? What is the impact of allowing approximations?

$R(M_2) = \underline{R}(M_2) = 7$

A part of the 2×2 matrix multiplication tensor:

$$\begin{aligned} t \quad := \quad & x_{0,0} \otimes x_{0,0} \otimes x_{0,0} \\ & + x_{0,1} \otimes x_{1,0} \otimes x_{0,0} \\ & + x_{0,1} \otimes x_{1,1} \otimes x_{1,0} \\ & + x_{0,0} \otimes x_{0,1} \otimes x_{1,0} \\ & + x_{1,0} \otimes x_{0,0} \otimes x_{0,1} \\ & + x_{1,0} \otimes x_{0,1} \otimes x_{1,1} \end{aligned}$$

$R(t) = 6$, but $\underline{R}(t) = 5$:

$$\begin{aligned} \varepsilon \cdot t \quad := \quad & (x_{0,1} + \varepsilon x_{0,0}) \otimes (x_{0,1} + \varepsilon x_{1,1}) \otimes x_{1,0} \\ & + (x_{1,0} + \varepsilon x_{0,0}) \otimes x_{0,0} \otimes (x_{0,0} + \varepsilon x_{0,1}) \\ & - x_{0,1} \otimes x_{0,1} \otimes (x_{0,0} + x_{1,0} + \varepsilon x_{1,1}) \\ & - x_{1,0} \otimes (x_{0,0} + x_{(0,1)} + \varepsilon x_{1,0}) \otimes x_{0,0} \\ & + (x_{0,1} + x_{1,0}) \otimes (x_{0,1} + \varepsilon x_{1,0}) \otimes (x_{0,0} + \varepsilon x_{1,1}) \end{aligned}$$

The matrix multiplication exponent ω is not affected by approximations [Bini 1980]:

$$\begin{aligned}
\omega &= \lim_{n\to\infty} \log_n(R(M_n)) \\
&= \lim_{n\to\infty} \log_n(\mathrm{WR}(sM_n)) \\
&= \lim_{n\to\infty} \log_n(\mathrm{CR}(sM_n)) \\
&= \lim_{n\to\infty} \log_n(\underline{R}(M_n)) \\
&= \lim_{n\to\infty} \log_n(\underline{\mathrm{WR}}(sM_n)) \\
&= \lim_{n\to\infty} \log_n(\underline{\mathrm{CR}}(sM_n))
\end{aligned}$$

Border rank upper bounds give fast matrix multiplication algorithms [Bini, Capovani, Romani, Lotti 1980].

$$\begin{array}{ccccc}
\mathrm{WR}(p) & \geq & \mathrm{CR}(p) & \geq & \mathrm{w}(p) \\
\rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} \\
\underline{\mathrm{WR}}(p) & \geq & \underline{\mathrm{CR}}(p) & \geq & \underline{\mathrm{w}}(p)
\end{array}$$

- For $p = x^2 y$ we have $\mathrm{WR}(p) = 3 > 2 = \underline{\mathrm{WR}}(p)$
- For $p = y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3 + x_1 x_2 x_3$ we have $\mathrm{CR}(p) > 2 = \underline{\mathrm{CR}}(p)$ [Hüttenhain 2017]
- For $p = (x_1 y_1 + \cdots x_8 y_8) z^{39}$ we have $\mathrm{w}(p) > 2 = \underline{\mathrm{w}}(p)$
  proved via combining [Allender-Wang 2015] and [Bringmann, I, Zuiddam 2017]

$$\begin{tikzcd}
\mathrm{WR}(p) \arrow[r, dash, "\geq"] \arrow[d, dash, "\geq"] & \mathrm{CR}(p) \arrow[r, dash, "\geq"] \arrow[d, dash, "\geq"] & \mathrm{w}(p) \arrow[d, dash, "\geq"] \\
\underline{\mathrm{WR}}(p) \arrow[r, dash, "\geq"] \arrow[urr, dash, "\geq"] & \underline{\mathrm{CR}}(p) \arrow[r, dash, "\geq"] & \underline{\mathrm{w}}(p)
\end{tikzcd}$$

Theorem (de-bordering) [Bläser, Dörfler, I CCC2021]

$\underline{\mathrm{WR}}(p) \geq \mathrm{w}(p)$.

Proof: Via $\mathrm{w}_{\otimes}$ and Nisan's flattenings ("tensor partial derivatives").

De-bordering is a recent topic in algebraic complexity theory.
New techniques in [Dutta Dwivedi Saxena CCC2021], [Dutta, Gesmundo, Ikenmeyer, Jindal, Lysikov 2022].

Linear forms $a_{i,j}$ and $b_{i,j}$. Matrix product with non-commuting entries:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,r} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,r} \end{pmatrix} \boxtimes \begin{pmatrix} b_{1,1} & \cdots & b_{1,r} \\ \vdots & \ddots & \vdots \\ b_{r,1} & \cdots & b_{r,r} \end{pmatrix} := \begin{pmatrix} \sum_{i=1}^r a_{1,i} \otimes b_{i,1} & \cdots & \sum_{i=1}^r a_{1,i} \otimes b_{i,r} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^r a_{r,i} \otimes b_{i,1} & \cdots & \sum_{i=1}^r a_{r,i} \otimes b_{i,r} \end{pmatrix}$$

Recall

$\mathrm{w}_{\otimes}(t)$ is the smallest r such that $\exists$ linear forms with $t =$

$$\begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

Theorem [Nisan 1991]

Consider the maps F_i: $\begin{array}{ccc} \bigotimes^d \mathbb{C}^n & \simeq & \mathsf{Mat}\left(\dim \bigotimes^i \mathbb{C}^n, \dim \bigotimes^{d-i} \mathbb{C}^n\right). \\ t & \mapsto & F_i(t) \end{array}$ We have $\forall t: \ \mathrm{w}_{\otimes}(t) = \max\{\mathsf{rank}(F_i(t))\}$.

Conclusion [Forbes 2016]

$\forall t: \ \underline{\mathrm{w}_{\otimes}}(t) = \mathrm{w}_{\otimes}(t)$.

Theorem (de-bordering) [Bläser, Dörfler, I CCC2021]

$\underline{\mathrm{WR}}(p) \geq \mathrm{w}(p)$.

Proof: Let $\pi : \bigotimes^d \mathbb{C}^n \to \mathbb{C}[x_1, \ldots, x_n]_d$.
Start with a border Waring rank decomposition

$$p = \lim_{\varepsilon\to 0} \begin{pmatrix} \ell_1(\varepsilon) & \ell_2(\varepsilon) & \cdots & \ell_r(\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \cdots \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} \ell_1(\varepsilon) \\ \vdots \\ \ell_r(\varepsilon) \end{pmatrix}$$

$$= \pi\left(\lim_{\varepsilon\to 0} \begin{pmatrix} \ell_1(\varepsilon) & \ell_2(\varepsilon) & \cdots & \ell_r(\varepsilon) \end{pmatrix} \boxtimes \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \boxtimes \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_1(\varepsilon) & 0 & 0 \\ & \ddots & \\ 0 & & 0 \\ 0 & 0 & \ell_r(\varepsilon) \end{pmatrix} \boxtimes \begin{pmatrix} \ell_1(\varepsilon) \\ \vdots \\ \ell_r(\varepsilon) \end{pmatrix} \right)$$

$$\overset{\text{Nisan}}{=} \pi\left(\begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix} \right)$$

$$\overset{\pi\ \text{hom.}}{=} \begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \cdot \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \cdot \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \cdots \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \cdot \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

$\square$

Remark: Computation via the trace is not closed

Recall:

$\mathrm{w}_\otimes(t)$ is the smallest r such that $\exists$ linear forms with $t =$

$$\begin{pmatrix} \ell_{1,1,1} & \ell_{1,2,1} & \cdots & \ell_{1,r,1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,3} & \cdots & \ell_{1,r,3} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,3} & \cdots & \ell_{r,r,3} \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_{1,1,d-1} & \cdots & \ell_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d-1} & \cdots & \ell_{r,r,d-1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,d} \\ \vdots \\ \ell_{1,r,d} \end{pmatrix}$$

Nisan: $\forall t$ we have $\underline{\mathrm{w}_\otimes}(t) = \mathrm{w}_\otimes(t)$.

Definition

For an order d tensor t the **trace complexity** $\mathrm{trw}_\otimes(t)$ is defined as the smallest r such that $\exists$ linear forms with $t =$

$$\mathsf{trace}\left(\begin{pmatrix} \ell_{1,1,1} & \cdots & \ell_{1,r,1} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,1} & \cdots & \ell_{r,r,1} \end{pmatrix} \boxtimes \begin{pmatrix} \ell_{1,1,2} & \cdots & \ell_{1,r,2} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,2} & \cdots & \ell_{r,r,2} \end{pmatrix} \boxtimes \cdots \boxtimes \begin{pmatrix} \ell_{1,1,d} & \cdots & \ell_{1,r,d} \\ \vdots & \ddots & \vdots \\ \ell_{r,1,d} & \cdots & \ell_{r,r,d} \end{pmatrix}\right)$$

Theorem [Bläser, I, Mahajan, Pandey, Saurabh CCC2020], confirming a conjecture of Forbes

$\exists t$ such that $\underline{\mathrm{trw}_\otimes}(t) < \mathrm{trw}_\otimes(t)$.

The highest known gap is just 1.

The gap is small, which can be seen by computing summands independently: $\forall t: \ \mathrm{w}_\otimes(t) \overset{!}{\leq} \big(\underline{\mathrm{trw}_\otimes}(t)\big)^2 \leq \big(\mathrm{w}_\otimes(t)\big)^2$.

Border complexity classes

Recall

VWaring is the set of p-families with polynomially bounded WR.

VChow is the set of p-families with polynomially bounded CR.

VBP is the set of p-families with polynomially bounded w.

Definitions ($\overline{\mathbf{VWaring}}$, $\overline{\mathbf{VChow}}$, $\overline{\mathbf{VBP}}$)

$\overline{\mathbf{VWaring}}$ is the set of p-families with polynomially bounded $\underline{\mathrm{WR}}$

$\overline{\mathbf{VChow}}$ is the set of p-families with polynomially bounded $\underline{\mathrm{CR}}$

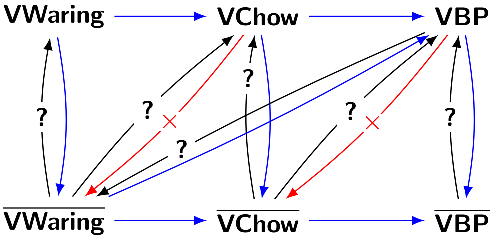
$\overline{\mathbf{VBP}}$ is the set of p-families with polynomially bounded $\underline{\mathrm{w}}$.

One can define a topology on the set of all p-families such that $\overline{\mathbf{VWaring}}$ is the closure of **VWaring** etc [I, Sanyal 2021].

$\boxed{\text{Valiant's conjecture 1979: } (\mathrm{per}) \notin \mathbf{VBP}}$ $\boxed{\text{Mulmuley-Sohoni's conjecture 2001: } (\mathrm{per}) \notin \overline{\mathbf{VBP}}}$

Is $\mathbf{VBP} = \overline{\mathbf{VBP}}$? This would imply that the questions at the heart of algebraic complexity theory are questions about algebraic geometry!

Notably we have no candidates for elements in $\overline{\mathbf{VWaring}} \setminus \mathbf{VWaring}$ or $\overline{\mathbf{VChow}} \setminus \mathbf{VChow}$ or $\overline{\mathbf{VBP}} \setminus \mathbf{VBP}$.
OPEN: Is $\overline{\mathbf{VChow}} \subseteq \mathbf{VNP}$?
Work in this direction by [Grochow Mulmuley Qiao 2016].



$A \longrightarrow B$ means $A \subseteq B$.

$A \overset{\times}{\longrightarrow} B$ means $A \not\subseteq B$.

- **VChow** $\not\subseteq$ $\overline{\textbf{VWaring}}$ via $\underline{\mathrm{WR}}(x_1 \cdots x_n) \geq \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq 2^{n/2}$ [Landsberg Teitler 2009]
- **VBP** $\not\subseteq$ $\overline{\textbf{VChow}}$ [Shpilka Wigderson 2001]

The open questions about this partially ordered set of 6 elements:

1. **VWaring** $\overset{?}{=}$ $\overline{\textbf{VWaring}}$
2. **VChow** $\overset{?}{=}$ $\overline{\textbf{VChow}}$
3. **VBP** $\overset{?}{=}$ $\overline{\textbf{VBP}}$
4. $\overline{\textbf{VWaring}}$ $\overset{?}{\subset}$ **VChow**
5. $\overline{\textbf{VChow}}$ $\overset{?}{\subset}$ **VBP**
6. $\overline{\textbf{VWaring}}$ $\overset{?}{=}$ **VBP**

Summary part "Closures, border complexity, and de-bordering"

- Border complexity has the advantage that we can rely on metapolynomials for lower bounds, i.e., they must exist
- In the matrix multiplication setting border complexity gives the same exponent ω.
- Open de-bordering questions: $\overline{\mathbf{VBP}} \overset{?}{\subseteq} \mathbf{VNP}$, even $\overline{\mathbf{VChow}} \overset{?}{\subseteq} \mathbf{VNP}$

We can move points around via base changes:

- For any 2×2 matrix A, define a new polynomial Ap via: $\quad (Ap)(\vec{x}) := p(A^t \vec{x})$

Example: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (ax^2 + bxy + cy^2) = cx^2 + bxy + ay^2$.

Orbit notation:

- $\mathsf{GL}_N p := \{gp \mid g \in \mathsf{GL}_N\}$.
- $\mathbb{C}^{N \times N} p := \{Ap \mid A \in \mathbb{C}^{N \times N}\}$.
- $\mathsf{GL}_N p \subseteq \mathbb{C}^{N \times N} p \subseteq \overline{\mathsf{GL}_N p} = \overline{\mathbb{C}^{N \times N} p}$.

Complexity described as orbit closures:

- $\underline{\mathrm{WR}}(p) \le r$ iff $p \in \overline{\mathsf{GL}_r(x_1^d + \cdots + x_r^d)}$
- $\underline{\mathrm{CR}}(p) \le r$ iff $p \in \overline{\mathsf{GL}_{rd}(x_{1,1} \cdots x_{1,d} + \cdots + x_{r,1} \cdots x_{r,d})}$
- $\underline{R}(t) \le r$ iff $t \in \overline{\mathsf{GL}_{3r}(x_{1,1} \otimes x_{2,1} \otimes x_{3,1} + \cdots + x_{1,r} \otimes x_{2,r} \otimes x_{3,r})}$
- $\underline{\mathrm{w}}(p) \le r$ iff $p \in \overline{\mathsf{GL}_{(d-2)r^2+2r} \mathrm{IMM}_r^{(d)}}$

$$\mathrm{IMM}_r^{(d)} := \begin{pmatrix} x_{1,1,1} & x_{1,2,1} & \cdots & x_{1,r,1} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1,2} & \cdots & x_{1,r,2} \\ \vdots & \ddots & \vdots \\ x_{r,1,2} & \cdots & x_{r,r,2} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1,3} & \cdots & x_{1,r,3} \\ \vdots & \ddots & \vdots \\ x_{r,1,3} & \cdots & x_{r,r,3} \end{pmatrix} \cdots \begin{pmatrix} x_{1,1,d-1} & \cdots & x_{1,r,d-1} \\ \vdots & \ddots & \vdots \\ x_{r,1,d-1} & \cdots & x_{r,r,d-1} \end{pmatrix} \cdot \begin{pmatrix} x_{1,1,d} \\ \vdots \\ x_{1,r,d} \end{pmatrix}$$

In general, the orbit closure containment problem is NP-hard, [Bläser, I, Jindal, Lysikov STOC 2018].

Group action on metapolynomials

$\mathbb{A} := \mathbb{C}[x,y]_2 = \langle x^2, xy, y^2 \rangle$. Every element in $\mathbb{A}$ can be represented as $ax^2 + bxy + cy^2$.

- $X := \{p \in \mathbb{A} \mid \exists \alpha, \beta \in \mathbb{C} : p = (\alpha x + \beta y)^2\}$ set of Waring rank 1 polynomials
- $p \in X$ iff $\Delta(p) = b^2 - 4ac = 0$.

Moving polynomials around

We can move polynomials around via base changes:

- For any 2×2 matrix A, define a new polynomial Ap via: $(Ap)(\vec{x}) := p(A^t \vec{x})$

Example: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (ax^2 + bxy + cy^2) = cx^2 + bxy + ay^2$.

Key observation: If $p \in X$, then $Ap \in X$.

This works in for all algebraic complexity measures, if we allow homogeneous linear inputs for free.

Moving metapolynomials around

- For any 2×2 matrix A, define a new polynomial $A\Delta$ via: $(A\Delta)(p) := \Delta(A^t p)$

Example: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A(b^2 - 4ac) = (Ab)^2 - 4(Aa)(Ac) = b^2 - 4ca = b^2 - 4ac$.

In general, one can calculate: $A(b^2 - 4ac) = \det(A)^2(b^2 - 4ac)$.

Highest weight metapolynomials

$\Delta := b^2 - 4ac$.
One can calculate: $A\Delta = \det(A)^2\Delta$.
In particular

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \Delta = \Delta; \qquad \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \Delta = \alpha_1^2 \alpha_2^2 \Delta$$

Thus Δ is a **highest weight metapolynomial of weight (2,2).**

Definition (highest weight metapolynomial, HWP)

A function Δ is called a **highest weight metapolynomial** of weight $\lambda = (\lambda_1, \ldots, \lambda_N)$, if

- Δ is invariant under the action of upper triangular matrices with 1s on the diagonal
- and Δ gets rescaled by $\alpha_1^{\lambda_1} \cdots \alpha_N^{\lambda_N}$ under the action of diagonal matrices $\mathrm{diag}(\alpha_1, \ldots, \alpha_N)$.

Complexity lower bounds via highest weight metapolynomials

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Recall: Want Δ vanishing on X and $\Delta(q) \neq 0$.

Theorem (representation theory)

If $q \notin X$, then there exists a HWP Δ such that $A\Delta$ vanishes on X and $(A\Delta)(q) \neq 0$ for a generic matrix A.

This works in high generality. We just need that X is closed under the action of GL_N.

Crucial conclusion

If complexity lower bounds exist, then there exist highest weight polynomials proving them.

For this conclusion we must allow approximations (border complexity) in our computational model.

Complexity of highest weight metapolynomials

Crucial conclusion

If complexity lower bounds exist, then there exist HWPs proving them.

Hauenstein-I-Landsberg 2013: Construct a degree 19 HWP that vanishes on all border rank ≤ 6 tensors, but not on M_2, hence $\underline{R}(M_2) \geq 7$. Bläser-Christandl-Zuiddam 2017 lift this bound to border support rank by working with this HWP.

Theorem (Garg, I, Makam, Oliveira, Walter, Wigderson, CCC 2020)

The hyperpfaffian, which is a HWP, is VNP-complete.

Theorem (Bläser, Dörfler, I, arXiv:2002.11594)

If HWPs are encoded by Young tableaux (which is very efficient), then it is NP-hard to evaluate them at a point of Waring rank 3. (Efficient evaluation is possible if the tableau has low treewidth.)

For even n, on $\mathbb{C}[x_1, \ldots, x_n]_n$ there exists a unique HWP Δ_n of weight $(n, n, \ldots, n)$. If for all even n we have $\Delta_n(x_1, \ldots, x_n) \neq 0$, then the Alon-Tarsi conjecture (1992) on Latin squares is true. This is only known for $n = p \pm 1$, so the first unknown case is $n = 26$ [Glynn 2010], [Drisko 1997].

Mulmuley and Sohoni's heuristic attempt: Occurrence Obstructions

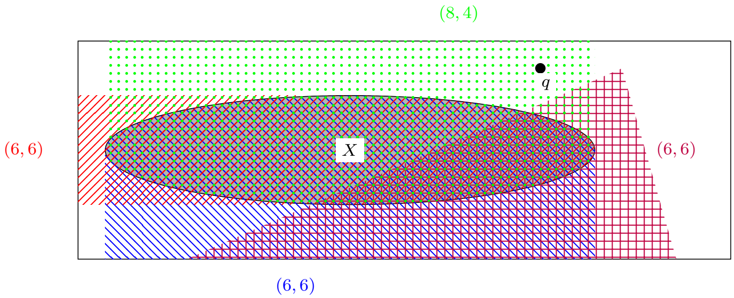
Consider the finite dimensional vector space of highest weight metapolynomials Δ of weight λ.

Proposition (a coarse technique for finding complexity lower bounds)

If there exists λ such that for a generic matrix A we have

- for **all (!)** HWPs Δ of weight λ: $A\Delta$ vanishes on X
- there exists a HWP Δ of weight λ such that $(A\Delta)(q) \neq 0$

then $q \notin X$.



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- there exists a HWP Δ of weight λ such that $(A\Delta)(q) \neq 0$

then $q \notin X$.

- We used this approach to show nontrivial border rank lower bounds for the matrix multiplication tensor [Bürgisser, I; STOC 2011, STOC 2013].
- Mulmuley and Sohoni conjectured that this approach could show superpolynomial lower bounds on $n(m)$ for $x_1^{n-m}\mathrm{per}_m \in \overline{\mathsf{GL}_{n^2}\det_n}$ This was too optimistic:

> In [I, Panova; FOCS 2016] and later [Bürgisser, I, Panova; FOCS 2016, JAMS] we prove that this approach **cannot** give $\mathrm{dc}(\mathrm{per}_m) > m^{25}$.

Remark: The padding plays a crucial role in the proof. For $\mathrm{w}(\mathrm{per})$ this might still be possible.

Summary part "Group actions and highest weights"

- Border complexity lower bounds questions can often be phrased via orbit closures.
- The group action lifts to the metapolynomials.
- If lower bounds exist, then they can be proved via Highest Weight Meta-polynomials (HWPs).
- Some HWPs are VNP-hard, and evaluation leads to difficult combinatorics.
- Occurrence obstructions are a coarser approach: **All** HWPs of a type must vanish.

Summary

- Border complexity and orbit closures have the advantage that we can study lower bounds via metapolynomials.
- In the matrix multiplication setting border complexity gives the same exponent ω.
- Open de-bordering questions: $\overline{\mathbf{VBP}} \overset{?}{\subseteq} \mathbf{VNP}$, even $\overline{\mathbf{VChow}} \overset{?}{\subseteq} \mathbf{VNP}$
- For any border complexity measure that is invariant under base changes:

separation via highest weight metapolynomials	**Always possible**, but hard in general.
separation via multiplicities	There is evidence that this might work. This is related to the coordinate rings of orbits.
separation via occurrences	Possible in some cases (matrix mult.), impossible in others (det vs padded per, and some finite homogeneous cases).

Thank you for your attention!