Parameterized algorithms

and

Fine-grained complexity

[[Part 1: Introduction to parameterized algorithms]]

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Parameterized Complexity

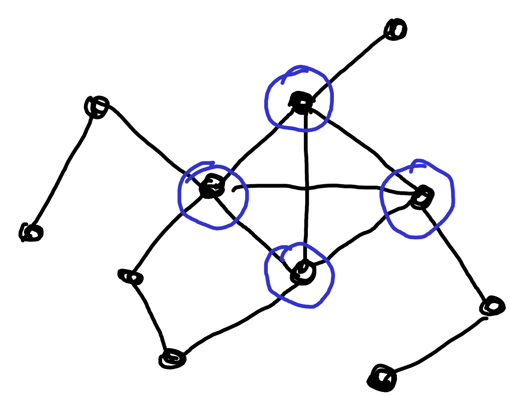
Classic complexity

Problem L: Input: $x \in \{0,1\}^*$

Question: Does $x \in L$?

Measure of hardness: $|x|$

Example: Does G have a clique / path / vertex cover ... of size k?



Intuition: Looking for a 10-clique is easier than for a 100-clique.

Parameterized Complexity

Each instance x supplied with parameter $k \in \mathbb{N}$.

- Possibly, parameterization $\kappa : \{0,1\}^* \to \mathbb{N}$, $k = \kappa(x)$.
- Also multiple parameters.

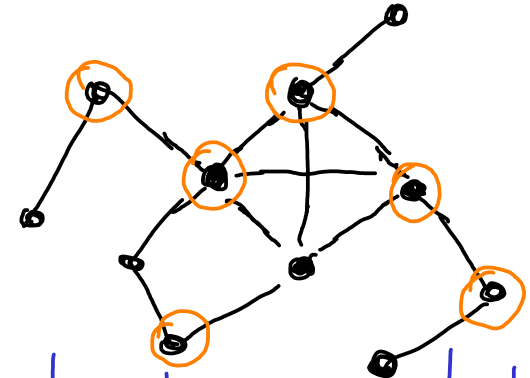
Idea Parameterized algorithm is efficient for small k.

Def Algorithm is XP (slice-wise polynomial) if runtime $f(k) \cdot |x|^{g(k)}$ (f, g computable)

Ex Clique in time $O(k^2 \cdot n^k)$

Def Algorithm is FPT (fixed-parameter tractable) if has runtime $f(k) \cdot |x|^c$ (f computable, c constant)

Vertex Cover



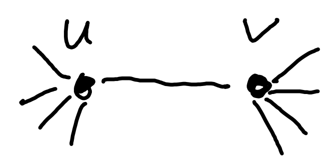
Input: G, $k \in \mathbb{N}$

Question: Is there $X \subseteq V(G)$, $|X| \leq k$, s.t. every edge has an endpoint in X?

Thm Vertex Cover can be solved in time $O(2^k \cdot (n+m))$.

Pf Recursion:

- Start with $X = \emptyset$, $b = k$
- If all edges covered: YAY!
- Otherwise uv not covered.
  - If $b = 0$: return
  - Otherwise branch.



[add u to X; b--] or [add v to X; b--]

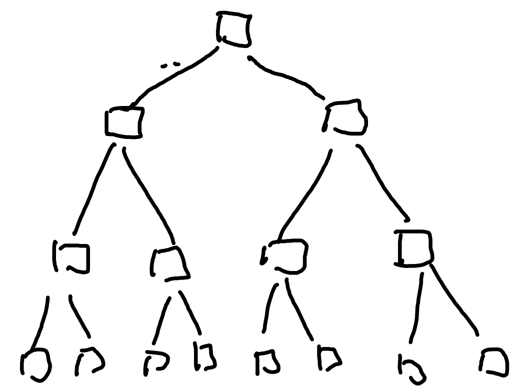
Vertex Cover

Analysis

- Recursion tree of depth $\leq k$, branching 2
- **Ergo**: $\leq 2^k$ leaves $\Rightarrow$ $\leq 2 \cdot 2^k - 1$ nodes.
- Every node needs $O(n+m)$ time.

$\rightarrow$ Time complexity $O(2^k \cdot (n+m))$

$\leq k$



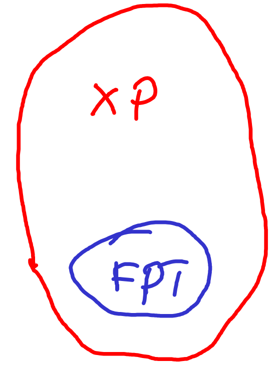
Cor: Vertex Cover parameterized by solution size k is fixed-parameter tractable.

Parameterized complexity, formally

<u>Def</u> Parameterized problem = Problem $Q \subseteq \{0,1\}^* \times \mathbb{N}$ (or param. $K : \{0,1\}^* \to \mathbb{N}$)

<u>Def</u> XP = Problems admitting an XP algorithm

FPT = Problems admitting an FPT algorithm



Which problems are XP?

Which problems are FPT?

How to prove lower bounds?

What running times can be achieved?

<u>Plan</u> (1) Overview of techniques of PA

(2) Lower bounds $\rightsquigarrow$ Fine-grained complexity

Examples

FPT: Vertex Cover/k

XP: Clique/k

↳ Clique/k is W[1]-hard

$\Rightarrow$ not FPT unless FPT = W[1]

para NP-hard: Coloring/# colors

↳ NP-hard for # colors = 3

Note Same problem with diff parameterization can have diff complexity

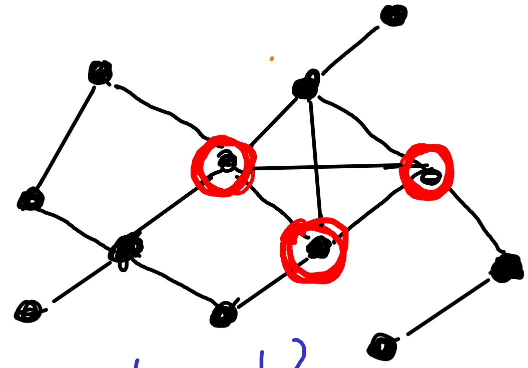
Example: Clique/Δ in time $O(2^{\Delta} \cdot n(n+m))$.

Parameterized complexity as an area

- Started in 90s (Downey, Fellows, Langston, ...)
  - ↳ Focused on complexity theory
- Boomed around 2005-2010
  - ↳ New algorithmic techniques
- Found applications in multiple areas:
  - inherent toolbox of param. techniques
  - way of thinking about parameters

- solution size
- structural parameters: Δ, tw, cw, tww, ...
- apx schemes: $1/\varepsilon$
- logic/databases: $\|\varphi\|$, arity
- strings: $|\Sigma|$, #mismatches, ...
- ILPs: #vars, structure
- CSPs
- Param. approximation

Feedback Vertex Set



Input: Graph G, $k \in \mathbb{N}$

Question: Is there $X \subseteq V(G)$, $|X| \leq k$, s.t. $G - X$ is a forest?

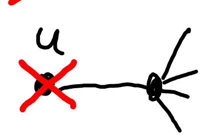
One of favorite problems: FPT by multiple techniques, none of which is trivial.

Thm FVS can be solved in time $O\big((5k)^k \cdot (n+m)\big)$

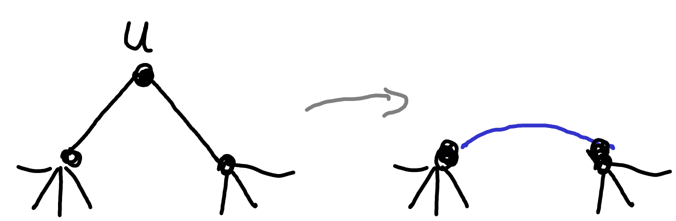
Pf **Idea:** Apply Reduction Rules

[R1] $\deg(u) = 0$ $\rightsquigarrow$ Delete u

[R2] $\deg(u) = 1$ $\rightsquigarrow$ Delete u



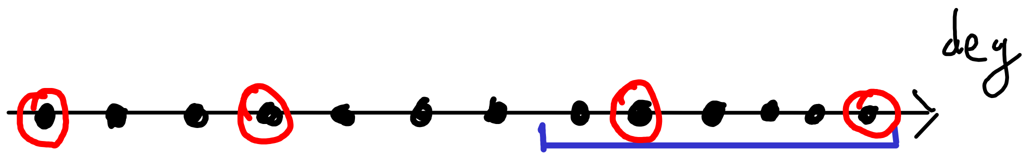
[R3] $\deg(u) = 2$ $\rightsquigarrow$ Dissolve u



FVS algorithm

Reduction rules exhaustively $\rightsquigarrow$ Multigraph with mindeg 3

Lemma If G is a multigraph with mindeg $\geqslant 3$, and G has a fvs X of size $\leqslant k$, then X contains one of $5k$ vertices of highest degree.

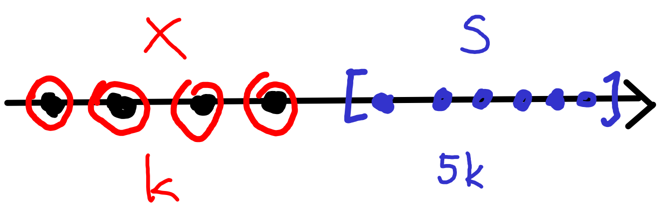


$S :=$ $5k$ highest

Pf $|E(G)| \geqslant \frac{3n}{2}$, $|E(G-X)| < |V(G-X)| \leq n$

$\hookrightarrow$ Deletion of X removed $> \frac{|E(G)|}{3}$ edges!

Suppose $S \cap X = \emptyset$. Then:



$$\sum_{u \in X} \deg(u) \leq \frac{1}{6} \sum_{u \in S \cup X} \deg(u) \leq \frac{1}{6} \cdot 2|E(G)| = \frac{|E(G)|}{3}$$

Contradiction $\lightning$ $\square$

FVS algorithm

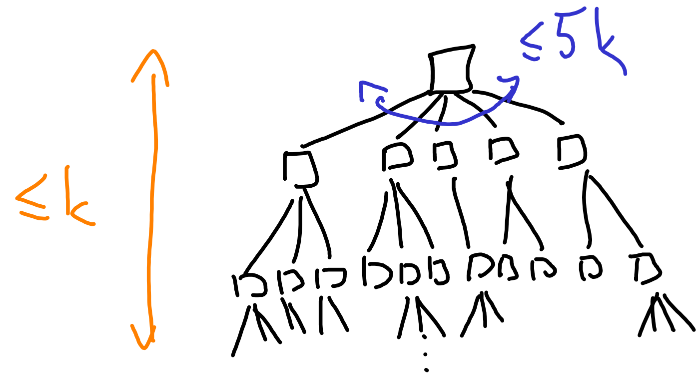
while $k>0$:

  apply reduction rules

  if $G=\emptyset$: YAY!

  $S := 5k$ highest degree vertices

  branch which $u\in S$ to delete



Analysis:

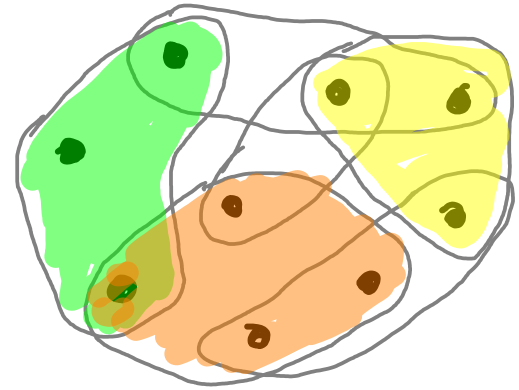
- recursion tree of depth $\leq k$, branching $\leq 5k$
- at most $2\cdot(5k)^k$ nodes
- each node uses time $O(n+m)$

→ runtime $O\left((5k)^k\cdot(n+m)\right)$ $\square$

Notes

- $5k \to 3k$ if one is careful.
- best known: $O(2.7^k\cdot n^c)$ [Li, Nederlof '20]

Set Cover



Input: Universe U, family $\mathcal{F} \subseteq 2^U$, $k \in \mathbb{N}$

Question: Is there $S \subseteq \mathcal{F}$ s.t. $|S| \leq k$ and $\bigcup S = U$?

Parameters

- k $\rightsquigarrow$ W[2]-complete, no FPT algo
- $|\mathcal{F}|$ $\rightsquigarrow$ brute-force $2^{|\mathcal{F}|} \cdot (|U| + |\mathcal{F}|)^{O(1)}$

Thm Set Cover can be solved in time $2^{|U|} \cdot (|U| + |\mathcal{F}|)^{O(1)}$

Idea DP on subsets of U

Set Cover algo

$\Phi[X] :=$ min # sets in $\mathcal{F}$ needed to cover X
X in $\mathcal{U}$

Need: $\Phi[\mathcal{U}]$

Have: $\Phi[\emptyset] = 0$

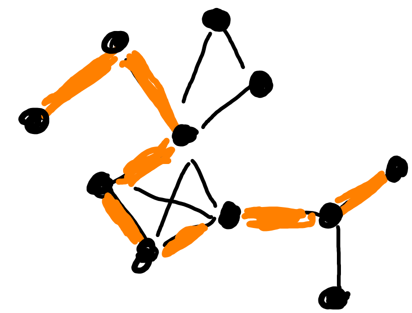
Recursion

$$\Phi[X] = 1 + \min_{\substack{A \in \mathcal{F} \\ A \cap X \neq \emptyset}} \Phi[X - A]$$

Just compute all $2^{|\mathcal{U}|}$ values, each in time $(|\mathcal{U}| + |\mathcal{F}|)^{O(1)}$ $\square$

Q: Is 2 in $2^{|\mathcal{U}|}$ optimal? Is it in $2^{|\mathcal{F}|}$?

k - Path



Input: Graph G, $k \in \mathbb{N}$

Question: Is there a simple path on k vertices in G?

Note: For $k = n$, this is Hamiltonian Path

Thm k-Path can be solved in $k^k \cdot n^{O(1)}$ randomized time.

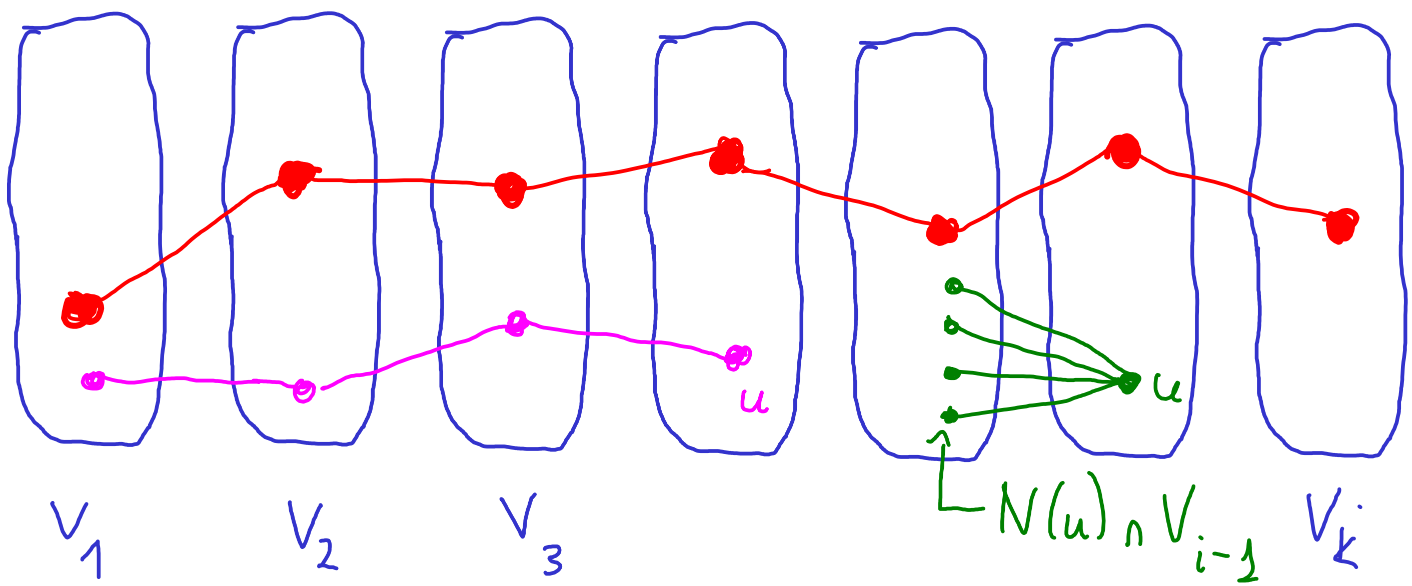
NO k-path $\Rightarrow$ Answer NO

EXISTS a k-path $\Rightarrow$ Answer YES with prob $\geq 1/2$

[Technique: Color Coding
Alon, Yuster, Zwick]

Colored k-Paths

Suppose $V(G)$ is colored into $V_1, V_2, V_3, \ldots, V_k$,
and we seek a path colored $1-2-3-\ldots-k$.



Claim This can be done in time $O(n+m)$.

Pf $\Phi[u]$ = Is there a $1-2-\ldots-i$ path ending at u? ($u \in V_i$)

$\Phi[u] = \text{true}$ ($u \in V_1$), $\quad \Phi[u] = \bigvee_{v \in N(u) \cap V_{i-1}} \Phi[v]$ ($u \in V_i$), $\quad$ return $\bigvee_{u \in V_k} \Phi[u]$ $\square$

k - Paths

Q How to achieve a k-colored instance?

Idea Draw a coloring at random!

$\lambda : V(G) \to \{1, \dots, k\}$ drawn uniformly, independently at random

$$V_i := \lambda^{-1}(i)$$

Suppose there is a solution $P = \overset{u_1}{\bullet} - \overset{u_2}{\bullet} - \overset{u_3}{\bullet} - \bullet - \bullet - \overset{u_k}{\bullet}$

$$\mathbb{P}[\, P \text{ is colored } 1\text{-}2\text{-}3\text{-}\dots\text{-}k \,] = \frac{1}{k^k}$$

Algorithm

- Draw λ at random
- Verify if there is a colored path in time $O(n+m)$

↳ Succeeds with probability $\geq 1/k^k$

k-Paths

Boosting probability: Repeat k^k times

NO k-path ⇝ Answer NO for sure

EXISTS a k-path ⇝ Failure to detect with probability

$$\leq \left(1-\frac{1}{k^k}\right)^{k^k} \leq e^{-k^k \cdot \frac{1}{k^k}} = e^{-1} < 1/2$$

$1-x \leq e^{-x}$

Running time: $O(k^k \cdot (n+m))$

What happened?

- k-Path is about disjointness constraints
- random partition of $V(G)$ makes [disjointness] ⇝ [colored]

Faster k-Path

Idea: Look for injectively colored k-path

Def $\lambda : V(G) \to \{1, \dots, k\}$

k-path P is colorful if λ is injective on $V(P)$

$$3 - 5 - 1 - 2 - 6 - 4$$

Stirling formula

$$\mathbb{P}[P \text{ becomes colorful under } \lambda] = \frac{k!}{k^k} \geq e^{-k}$$

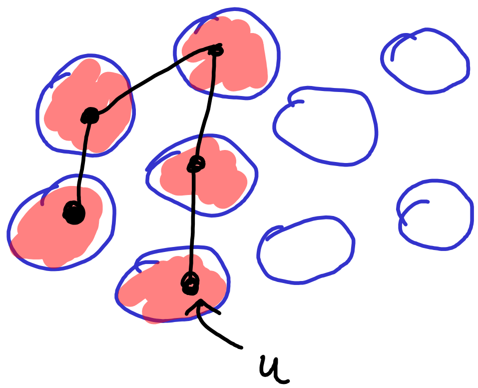
$\Rightarrow$ Only need to repeat the experiment e^k times.

[[Need: How to find a colorful path?]]

Idea: DP on subsets of colors

Colorful k-Paths

$$\Phi[\underset{\overset{\cap}{V_i}}{u}, \underset{\overset{\cup}{\{1,\dots,k\}}}{\overset{\cap}{X}}] = \text{Is there a colorful path using color } X \text{ that ends at } u\text{?}$$



X

Time complexity:

$$O((n \cdot 2^k) \cdot (n+m)) = 2^k \cdot n^{O(1)}$$

$$\Phi[\underset{\overset{\cap}{V_i}}{u}, \{i\}] = \text{true}$$

$$\Phi[\underset{\overset{\cap}{V_i}}{u}, X] = \bigvee_{v \in N(u)} \Phi[v, X - \{i\}]$$

Total runtime:

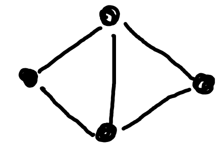
$$e^k \cdot 2^k \cdot n^{O(1)} = (2e)^k \cdot n^{O(1)}$$

Return $\bigvee_{u \in V(G)} \Phi[u, \{1, \dots, k\}]$

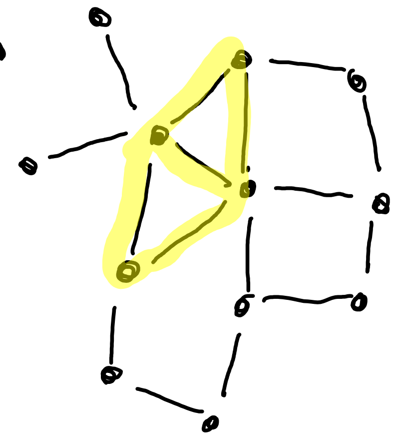
Dir: $2^k \cdot n^{O(1)}$, Undir: $1.66^k \cdot n^{O(1)}$

Subgraph Isomorphism

Input: Graphs H and G, $|V(H)| = k$



Question: Is H a subgraph of G?



$H = P_k \rightsquigarrow$ k-Path $\rightsquigarrow$ FPT

$H = K_k \rightsquigarrow$ k-Clique $\rightsquigarrow$ W[1]-hard

Thm SI can be solved in randomized time $k! \cdot \Delta^{O(k)} \cdot n^{O(1)}$ on graphs with maximum degree Δ.

$\hookrightarrow$ FPT parameterized by k and Δ $\longrightarrow$ Follows also from FO model-checking

For now: assume G is connected.

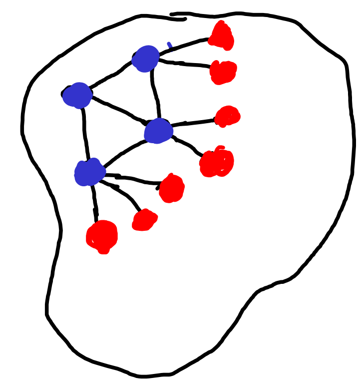
Subgraph Isomorphism

Randomly color $V(G)$ blue and red

↳ every vertex independently blue with prob $1/\Delta$

red with prob $1 - 1/\Delta$

Fix $A \subseteq V(G)$ s.t. there is an H-subgraph on A.

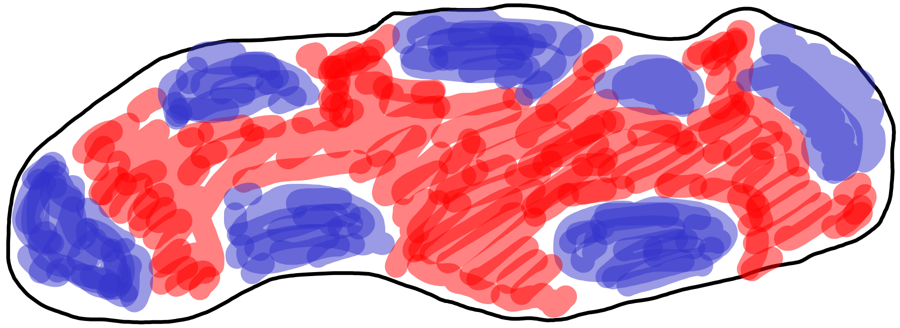


Event: A gets blue, and
$N(A)$ gets red.

$$\mathbb{P}[\text{Event}] \geq \left(\frac{1}{\Delta}\right)^{k} \cdot \left(1 - \frac{1}{\Delta}\right)^{\Delta k} \geq (\Delta e)^{-k}$$

Subgraph Isomorphism

Finding a solution in case of Event.



Go through blue connected components one by one

↳ For each of size k, check if H is a subgraph in time $O(k!)$

Total time:

repetitions

$$O(k!) \cdot (\Delta e)^k \cdot n^{O(1)}$$

one experiment

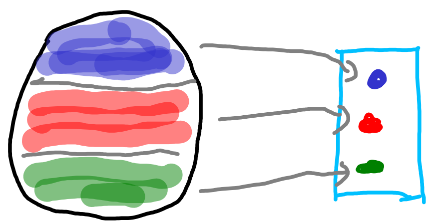
Disconnected H:

Solve a matching problem

comps of H $\longleftrightarrow$ blue comps

Derandomization

Algorithms obtained through Color Coding can be typically derandomized using universal hash families.



<u>Thm</u> There is a family $\mathcal{F}$ of functions $\{1, \ldots, n\} \to \{1, \ldots, k\}$ of size $e^k \cdot k^{O(\log k)} \cdot \log n$ such that for each $A \subseteq \{1, \ldots, n\}$ of size k, there exists $f \in \mathcal{F}$ that is injective on A.

$[\![$ Instead of sampling λ, check all $\lambda \in \mathcal{F}$. $]\!]$